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# Conformally Invariant Klein-Gordon Equation in Kaluza-Klein Theory

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Liu and Wesson discussed the Klein-Gordon scalar equation in a 5D manifold. We generalize Liu's discussions from the minimally coupled case to the conformally coupled case and discuss further the variation ratio of a particle mass.

## 1. INTRODUCTION

In order to unify gravity and electromagnetism, Kaluza and Klein generalized Einstein's general relativity to 5D mainfold (Bailin and Love, 1987). Although Kaluza-Klein theory hasn't succeeded, it gives us useful enlightenment in generalizing some physical theories in 4D to high dimension to obtain some useful physical conclusions. It has been developed and applied in many aspects (Ma, 1995). All kinds of high-dimensional theories developed for different aims are called Kaluza-Klein theory.

In the 1980s, Wesson proposed a new variable gravity, namely, a 5D space-time-mass theory (Wesson, 1984). Using dimensional analysis, Wesson introduced the fifth coordinate $x^4 = Gm/c^2$ ($c$ is velocity of light, $G$ is Newton gravitational constant, and $m$ is the rest mass) besides the 4D space–time coordinates. Thus matter itself is brought into a geometrical formalism. In this theory, the test mass of typical particle may change with time and position; this embodies the spirit of Mach's principle in the sense that inertial mass depends on the distribution of matter in the universe (Ma, 1990). Space-time-mass theory has been studied in great depth (Ma, 1990a, 1991; Overduin and Wesson, 1997; Wesson, 1999). Besides this way of realizing variable gravity by the extra dimension, Liu and Wesson recently generalized the 4D Klein-Gordon equation to 5D and realized variable gravity

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(variable mass) by the coupling between the scalar field and the gravitational field (Liu and Wesson, 2000). Their work was based merely on the minimally coupled case, and didn't discuss in detail theoretical predictions and astrophysical observations. In consideration of the special significance of conformal invariance in theoretical physics (Birrell and Davies, 1982; Nazlikaz and Padmanabhan, 1983; Wald, 1984), the present paper generalizes the minimally coupled case to the conformally coupled case, discusses the variable mass and compares the theoretical value with observational data of astrophysics.

## 2. GENERALIZATION OF KLEIN-GORDON EQUATION IN 5D

First, generalize the Klein-Gordon equation in flat space–time to curved space–time in a straightforward way. We use natural units in which $c = h = 1$ and let lower-case Greek letters run 0, 1, 2, 3 (for time and space). In 4D flat space–time, the conventional Klein-Gordon equation is

$$\eta^{\alpha\beta}\psi_{,\alpha\beta} + m_0^2\psi = 0 \tag{1}$$

Here $\eta^{\alpha\beta}$ = diagonal $(+1, -1, -1, -1)$ is the matric of Minkowski space, $m_0$ is the mass of a particle, $\psi$ is the 4D wave function. Equation (1) may be obtained from the Lagrangian density

$$l(x) = \frac{1}{2}\left(\eta^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta} - m_0^2\psi^2\right) \tag{2}$$

by constructing the action $S = \int l(x)\, d^4x$ and demanding $\delta S = 0$. Formally, the scalar field equation in curved space–time proceeds in close analogy to the Minkowski space case. The Lagrangian density is

$$l(x) = \frac{1}{2}[-g(x)]^{1/2}\left\{g^{\alpha\beta}(x)\psi(x)_{,\alpha}\psi(x)_{,\beta} - \left[m_0^2 + \xi R(x)\right]\psi^2(x)\right\}. \tag{3}$$

Here $\xi$ is a numerical factor and $R(x)$ is the Ricci scalar curvature. The coupling between the scalar field and the *gravitational* field is represented by the term $\xi R\psi^2$. The scalar field equation is

$$\psi^{\alpha}_{;\alpha} + m_0^2\psi + \xi R^{(4)}\psi = 0, \tag{4}$$

where $R^{(4)}$ is the 4D scalar curvature. Two value of $\xi$ are of particular interest: the minimally coupled case, $\xi = 0$, and the conformally coupled case, $\xi = 1/4$ $[(n-2)/(n-1)]$. Here $n$ is the space–time dimension.

Second, generalize the conformally coupled Klein-Gordon equation to 5D. Liu and Wesson generalized the 4D Klein-Gordon equation to 5D for the minimally coupled case. They pointed out that there are two obvious candidates for 5D generalization of (4), namely (Liu and Wesson, 2000)

$$\Psi^{A}_{;A} + m_0^2\Psi = 0 \tag{5}$$

$$\Psi^{A}_{;A} = 0 \tag{6}$$

where upper-case Latin letters run $0,\ 1,\ 2,\ 3,\ 4$, and $\Psi^A = g^{AB}\Psi_{,B}$. However they preferred (6) to (5) in order to obtain variable mass by hiding the mass in the extra dimension. Considering the conformally coupled case besides their arguments, we choose the Klein-Gordon equation in 5D Riemannian space

$$\Psi^{A}_{;A} + \xi R^{(5)}\Psi = 0 \tag{7}$$

where $R^{(5)}$ is the 5D scalar curvature and $\xi = 1/4[(n-2)/(n-1)] = 3/16$, $n = 5$.

## 3. GENERAL EXPRESSION OF A PARTICLE MASS

In this section we reduce the 5D Klein-Gordon equation (7) to the massive 4D Klein-Gordon equation (4) and thereby obtain a general expression for the mass $m$ of a test particle. Consider the line element

$$dS^2 = g_{AB}\,dx^A\,dx^B = g_{\alpha\beta}\,dx^\alpha\,dx^\beta - \phi^2\,dl^2 \tag{8}$$

Here $g_{\alpha\beta}$ is the metric of 4D Riemannian space, $g_{\alpha\beta} = g_{\alpha\beta}(x^\alpha, l)$ and $\phi = \phi(x^\alpha, l)$. Introducing the 5D Christoffel simbol $\Gamma^{C}_{AB}$ and the covariant derivative, we have

$$\begin{aligned}\Psi^{A}_{;A} + \frac{3}{16}R^{(5)}\Psi &= g^{AB}\left(\Psi_{,AB} - \Gamma^{C}_{AB}\Psi_{,c}\right) + \frac{3}{16}g^{AB}R_{AB}\Psi \\ &= g^{\alpha\beta}\left(\Psi_{,\alpha\beta} - \Gamma^{\lambda}_{\alpha\beta}\Psi_{,\lambda}\right) + \frac{1}{2}g^{\alpha\beta}g^{44}g_{\alpha\beta,4}\Psi_{,4} + g^{44}\Psi_{,44} \\ &\quad + \frac{1}{2}g^{44}g^{\lambda\alpha}g_{44,\alpha}\Psi_{,\lambda} - \frac{1}{2}g^{44}g^{44}g_{44,4}\Psi_{,4} + \frac{3}{16}R^{(4)}\Psi \\ &\quad + \frac{3}{16}g^{\alpha\beta}R^{4}_{\alpha 4\beta}\Psi + \frac{3}{16}g^{44}R^{\alpha}_{4\alpha 4}\Psi\end{aligned} \tag{9}$$

However, the first term on the right-hand side is $\Psi^{\alpha}_{;\alpha}$. From (7) we obtain the 4D Klein-Gordon equation with extra terms derived from the fifth dimension that define an effective 4D mass

$$\Psi^{\alpha}_{;\alpha} + m^2\Psi + \frac{1}{6}R^{(4)}\Psi = 0 \tag{10}$$

$$m^2 \equiv \phi^{-2}\Psi^{-1}\left[\phi g^{\alpha\beta}\phi_{,\alpha}\Psi_{,\beta} + \left(\phi^{-1}\overset{*}{\phi} - \frac{1}{2}g^{\alpha\beta}\overset{*}{g}_{\alpha\beta}\right)\overset{*}{\Psi} - \overset{**}{\Psi}\right] + \frac{1}{48}R^{(4)} + \frac{3}{16}g^{\alpha\beta}R^{4}_{\alpha 4\beta} - \frac{3}{16}\phi^{-2}R^{\alpha}_{4\alpha 4} \tag{11}$$

Here an overstar denotes the partial derivative with respect to $x^4 = l$. Expression (11) expresses the effective 4D mass with 5D wave function $\Psi$ and the coefficient of the metric. In general the mass $m = m(x^\alpha,\ l)$ is variable, which depends on both the space–time coordinates and the extra coordinate. In order to gain the constant rest mass in Minkovski space ($g^{\alpha\beta} = \eta^{\alpha\beta},\ \phi = 1$), we may restrict the wave function $\Psi : \Psi = \Psi(x^\alpha)\ e^{im_0 l}$. This indicates that the scalar field $\Psi$ still depends on the fifth coordinate $l$ by the phase factor $e^{im_0 l}$.

## 4. TWO TYPICAL EXAMPLES

Consider the two most significant solutions in gravity: the static and spherically symmetric solution, and the Friedman cosmological solution. The most important and the most persuasive examinations in general relativity have also been carried out under the two circumstances, and have been supported by experiments and observational data on astrophysics. Therefore, any generalization of general relativity should first be examined under these circumstances. We will take the two cases for examples, but employ a new approach slightly different from that employed by Liu and Wesson.

First, consider the well-known line element describing the static and spherically symmetric gravitational field (Wesson, 1999).

$$\begin{aligned} dS^2 &= A^a\, dt^2 - A^{-a-b}\, dr^2 - A^{1-a-b}r^2\, d\Omega^2 - A^b\, dl^2 \\ A(r) &= 1 - 2\mu G/r \\ 1 &= a^2 + ab + b^2 \end{aligned} \tag{12}$$

Here $\mu$ and one of $a$ or $b$ are parameters, which are determined by observations. The case $a = 1,\ b = 0$ reduces (12) to the 4D Schwarzschild solution plus an extra dimension, in which the mass of a particle $m = m_0$ is constant (This conclusion is obvious because Einstein theory is an invariable gravity), $\mu$ is the central mass, and Kaluza-Klein theory is degenerated into Einstein theory. It implies that 4D Einstein theory with matter is embedded in 5D Kaluza-Klein theory for vacuum. In order to gain the expression for the mass of a test particle, we adopt the approximate solution $\Psi(x, l) = A^\omega\ e^{im_0(l-t)}$, where $\omega$ is a parameter, namely the scalar field wave function $\Psi$ is viewed as a plane wave propagating in the fifth coordinate, and describes a spin-0 particle in the static and spherically symmetric gravitational field. Since we are mainly interested in properties of the Klien-Gordon equation in weak field, it can be verified that the wave function $\Psi$ satisfies Eq. (7) in first approximation. The mass by (11) and (12) is

$$m^2 = A^{-b}m_0^2 - 2br^{-2}A^{a+b-2}\left(\frac{\mu G}{r}\right)^2 + \frac{1}{48}R^{(4)} + \frac{3}{16}g^{\alpha\beta}R^4_{\alpha 4\beta} - \frac{3}{16}A^{-b}R^{\alpha}_{4\alpha 4} \tag{13}$$

We find that the mass $m = m(r)$ may vary with $r$ in the static and spherically symmetric gravitational field, and $m(\infty) = m_0$, which is expected by Mach's principle. To calculate the variation ratio of mass with respect to $r$, we can estimate the value of $b$ and discuss the physical meaning of the constant $\mu$ by using the result of the post-Newton test in the 4D theory, and then obtain the specific expression of the mass. We carry out the well-known coordinate transformation

$$\rho = \frac{1}{2}[r - \mu G + \sqrt{r(r - 2\mu G)}] \tag{14}$$

The line element in an isotropic coordinate system can be written as (Dauidson and Owen, 1985; Gross and Perry, 1983)

$$dS^2 = \left(\frac{1-\mu G/2\rho}{1+\mu G/2\rho}\right)^{2a} dt^2 - \left(\frac{1-\mu G/2\rho}{1+\mu G/2\rho}\right)^{2(1-a-b)} \times\left(1+\frac{\mu G}{2\rho}\right)^4[d\rho^2 + \rho^2\, d\Omega^2] - \left(\frac{1-\mu G/2\rho}{1+\mu G/2\rho}\right)^{2b} dl^2 \tag{15}$$

We carry out the Robertson expansions of the above coefficients, which are

$$\begin{aligned}\left(\frac{1-\mu G/2\rho}{1+\mu G/2\rho}\right)^{2a} &= 1 - 2a\frac{\mu G}{\rho} + 2a^2\left(\frac{\mu G}{\rho}\right)^2 + \cdots \\ &= 1 - 2\alpha\frac{MG}{\rho} + 2\beta\left(\frac{MG}{\rho}\right)^2 + \cdots \\ \left(\frac{1-\mu G/2\rho}{1+\mu G/2\rho}\right)^{2(1-a-b)}\left(1+\frac{\mu G}{2\rho}\right)^4 &= 1 + 2(a+b)\frac{\mu G}{\rho} + \cdots \\ &= 1 + 2\gamma\frac{MG}{\rho} + \cdots\end{aligned} \tag{16}$$

where $\alpha = a\mu/M$, $\beta = (a\mu/M)^2$, and $\gamma = (a+b)\mu/M$ are three post-Newtonian parameters, and $M$ is the mass of the source of the gravitational field. According to the definition of $M$, we should set $\alpha = 1$. Then we have $\beta = 1$, $\gamma = 1 + b/a$.

The 4D Einstein theory gives $\alpha = \beta = \gamma = 1$, which corresponds to the case $b \to 0$ and $a \to 1$. The experimental result of the "time delay" of radar signals gives $\gamma = 1.000 \pm 0.002$ (Reasenberg *et al.*, 1979). We take the upper limit of the value of $\gamma$ ($b/a \approx 0.002$) and have $b = 0.002$ and $a = 0.999$ or $b = -0.002$ and $a = -0.999$. We should point out that the solution with $b = 0.002$ slightly diverges from the vacuum field equation. This can be explained by the fact that the energy–momentum tensor of the scalar field is also the part of the source of the gravitational field.

To check if the result is reasonable, consider how the gravitational field on the surface of the earth affects a neutral $\pi$ meson. Putting $b = \pm 0.002$, $a = \pm 0.999$, $M = 5.977 \times 10^{24}$ kg, $r = 6.370 \times 10^6$ m, and $m_0 = 2.405 \times 10^{-28}$ kg into the expression (13) gives mass $m \approx [1 + b\mu G/(c^2 r)]\ m_0 \approx [1 \pm 1.39 \times 10^{-12}]\ m_0$, which is hardly different from $m_0$ ($= m(\infty)$) and does not conflict with the present observational data.

Second, we merely discuss the standard Friedman model with $k = 0$ in detail. Consider the following class of cosmological solution, which seems like a plane wave propagating in the fifth dimension (Liu and Wesson, 1994).

$$dS^2 = A^{-(1+3\gamma)}\, dt^2 - A^2(dr^2 + r^2\, d\Omega^2) - A^{-(1+3\gamma)}\, dl^2$$
$$A = (Hu)^{\frac{1}{2+3r}}$$
$$p = \gamma\rho \qquad 8\pi\rho G = \frac{3H^2}{(2+3\gamma)^2 A^{3(1+\gamma)}} \tag{17}$$

Here $H = \alpha(2 + 3\gamma)$ is a parameter with physical dimensions of $T^{-1}$ or $L^{-1}$ and $u = t - l$. For a free particle, we still write $\Psi(x, l) = A^{\omega} e^{im_0(l-t)}$, where $\omega$ is a determinable parameter. Coupling (11) with (17) gives the mass of a test particle

$$m^2 = A^{(1+3\gamma)} \left\{ \left[ m_0^2 - \frac{(7+3\gamma)\dot{A}\dot{A}^{\omega}}{2AA^{\omega}} - \frac{\ddot{A}^{\omega}}{A^{\omega}} \right] + im_0 \left[ \frac{(7+3\gamma)\dot{A}}{2A} + \frac{2\dot{A}^{\omega}}{A^{\omega}} \right] \right\}$$
$$+ \frac{1}{48} R^{(4)} + \frac{3}{16} g^{\alpha\beta} R^4_{\alpha 4\beta} - \frac{3}{16} A^{(1+3\gamma)} R^{\alpha}_{4\alpha 4} \tag{18}$$

Here an overdot denotes the derivative with respect to $u$. To make $m^2$ real, we set the second term on the right-hand side to zero, giving us $\omega = -(7 + 3\gamma)/4$. Then finally (18) becomes

$$m^2 = (Hu)^{\frac{1+3\gamma}{2+3\gamma}} \left[ m_0^2 - \frac{(7+3\gamma)(1+9\gamma)}{16(2+3\gamma)^2} (Hu)^{-2} H^2 \right]$$
$$+ \frac{1}{48} R^{(4)} + \frac{3}{16} g^{\alpha\beta} R^4_{\alpha 4\beta} - \frac{3}{16} (Hu)^{\frac{1+3\gamma}{2+3\gamma}} R^{\alpha}_{4\alpha 4} \tag{19}$$

So for late times we have $(t \gg l) u \approx t$. Carrying out the coordinate transformation

$$Ht = (H_0 T)^{\frac{2(2+3\gamma)}{3(1+\gamma)}} \quad H_0 = \frac{3(1+\gamma)}{2(2+3\gamma)} H \tag{20}$$

We have the line element in the comoving coordinate system

$$dS^2 = dT^2 - R^2(dr^2 + r^2\, d\Omega^2) - \phi^2\, dl^2$$
$$R = (H_0 T)^{\frac{2}{3(1+\gamma)}} \quad \phi = (H_0 T)^{-\frac{(1+3\gamma)}{3(1+\gamma)}} \tag{21}$$

Here $T$ is the proper time of the universe. The mass is

$$m^2 = (H_0 T)^{\frac{2(1+3\gamma)}{3(1+\gamma)}} \left[ m_0^2 - \frac{(7+3\gamma)(1+9\gamma)}{36(1+\gamma)^2} (H_0 T)^{-\frac{4(2+3\gamma)}{3(1+\gamma)}} H_0^2 \right] + \frac{1}{48} R^{(4)} + \frac{3}{16} g^{\alpha\beta} R^4_{\alpha 4\beta} - \frac{3}{16} (H_0 T)^{\frac{2(1+3\gamma)}{3(1+\gamma)}} R^\alpha_{4\alpha 4} \tag{22}$$

Then we will discuss in detail the two special cases: the later universe ($\gamma = 0$) and the early universe ($\gamma = 1/3$), and calculate the variation ratio of the mass.

For the later (dust) universe ($\gamma = 0$), the line element and the mass are

$$dS^2 \approx dT^2 - (TH_0)^{4/3}(dr^2 + r^2\, d\Omega^2) - (TH_0)^{-2/3}\, dl^2 \tag{23}$$

$$m^2 = (H_0 T)^{2/3} m_0^2 - \frac{7}{36} (H_0 T)^{-2/3} H_0^2 - \frac{1}{18} T^{-2} \tag{24}$$

For the early (radiation-dominated) universe ($\gamma = 1/3$), the line element and the mass are

$$dS^2 \approx dT^2 - (TH_0)(dr^2 + r^2\, d\Omega^2) - (TH_0)^{-1}\, dl^2 \tag{25}$$

$$m^2 = (H_0 T) m_0^2 - \frac{1}{2} (H_0 T)^{-2} H_0^2 \tag{26}$$

Obviously, in both the cases the mass of a particle may change with the proper time of the universe $T$, which embodies the spirit of Mach's principle. In order to calculate the persent value of the mass variation and compare it with the observational data we put the age of the universe $T \sim 10^{10}$ year and Hubble's constant $H_0 \sim (10^{10}\ \text{year})^{-1}$. In the matter-dominated age, the variation ratio of a spin-0 neutral $\pi$ meson ($m_0 = 2.405 \times 10^{-28}$ kg) is $(dm/dT)/m \approx 3.333 \times 10^{-11}\ \text{year}^{-1}$, which is in agreement with the result calculated by Shapiro *et al.* (1971) and does not conflict with the observations.

## 5. CONCLUSION

We have built the 5D conformally invariable Klein-Gordon equation, which can be reduced to the normal 4D space–time to obtain the effective 4D variable mass of a test particle. We check this in detial for the cases of the static and spherically symmetric gravitational field and the Friedman universe. The results don't conflict with both experimental data in weak-field approximation and the present variation ratio of a particle mass. These results are very interesting from Mach's point of view. In the minimally coupled case, the Klein-Gordon equation also gives the general conclusion of the variable mass (Liu and Wesson, 2000); however, it is constant for the Schwarzschild and late-universe cases, which does not embody the spirit of Mach's principle. The 5D conformally invariable scalar field equation makes up the cases. Perhaps the conformally invariable Klein-Gordon equation in many generalized forms is the most ideal.

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